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Microscopic entropy of the most general four-dimensional BPS black hole

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Abstract: In a recent paper we have given the macroscopic and microscopic description of the generating solution of toroidally compactified string theory BPS black holes. In this note we compute its corresponding microscopic entropy. Since by definition the generating solution is the most general solution modulo U-duality transformations, this result accounts for the degeneracy of microstates determining the entropy of any regular BPS black holes of toroidally compactified string (or M) theory.

Keywords: D-branes, Black Holes in String Theory, Black Holes, Supersymmetry and Duality.

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Contents

1. Introduction

Recently we have given a description, both at macroscopic and microscopic level, of the generating solution of four-dimensional regular BPS black holes within toroidally compactified string (or M) theory, [1]. Acting by means of U-duality transformations on the above solution one can reconstruct any other solution of the same kind within the relevant four-dimensional effective theory (i.e. $N = 8$ supergravity) and also give for each of them a corresponding microscopic description in terms of bound states of stringy objects, [2]. Because of its very definition, the generating solution encodes the fundamental degrees of freedom (related to U-duality invariants) characterizing *the most general* regular BPS black hole within the same theory. Hence a detailed microscopic understanding of this solution is of considerable relevance for a deeper understanding of stringy oriented microscopic entropy counting.[1]

Although we have given a prediction for the microscopic entropy of the generating solution which is consistent with what is expected (see [6]), a missing aspect in our analysis was the *statistical* interpretation (and computation) of the predicted formula. In this note we fill such a lack. This can be done by suitably extending the microscopic computation of [7] (which was carried out in the context of Calabi-Yau compactifications) to the toroidal case. One of the main concerns in the analysis of [7] was to compute higher order corrections to the semi-classical result. In general one expects both α' and g_{string} corrections. It is known (see [7]–[10]) that the two affect the area law by a deformation of the effective horizon area and by a deviation from the area law itself. The latter has a leading term which is topological (originating

[1] For a review see for example [3, 4] and, more recently, [5] and references therein.

Brane	0	1	2	3	4	5	6	7	8	9	10
P_L	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$
M5	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$
M5	$\times$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$
M5 + $h_{(3)}$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\times$

Table 1: The M-theory configuration corresponding to the $N = 8,\ d = 4$ BPS black holes generating solution. The orientation along different WV directions (which has not been made explicit in this table) should be the suitable one so to preserve supersymmetry. We refer to [1] for details.

from R^4 terms in M-theory) and which thus depends on the particular compactifying manifold considered, giving therefore different contributions for Calabi-Yau and toroidal compactifications. Although our analysis is limited just to the semiclassical leading approximation we shall easily see that in the case of tori this topological term does not contribute.

The microscopic configuration corresponding to the generating solution presented in [1] can be described, in the type-IIA framework, as a bound state of 3 bunches of D4-branes intersecting on a point, a bunch of D0-branes on top of them plus some magnetic flux on the D4-branes world volume (in a way which is consistent with the residual supersymmetry of the solution) which induces D2 and (extra) D0-brane effective charges. Upon T-duality transformations along three internal directions the same solution is described in type IIB by four bunches of D3-branes intersecting at non-trivial SU(3) angles (where the non-trivial angle θ is in turn the T-dual of the magnetic flux). On the other hand, via a S-duality transformation, one may obtain the M-theory counterpart of the type-IIA system described above. In fact, in computing the statistical entropy, the M-theory picture is more convenient to deal with. Indeed, in this case the effective low-energy field theory describing the physics of the intersection is, in a suitable limit, a conformal 2-dimensional one, rather than conformal quantum mechanics, as it would be in the type-IIA setting. Even if the two pictures describe two S-dual regimes, the corresponding entropy, which is a U-duality invariant, is the same and so is the degeneracy of the corresponding microscopic degrees of freedom.

The M-theory configuration describing the generating solution is depicted in table 1 and consists of a set of three bunches of M5-branes intersecting on a (compact) line along which some units of KK-momentum has been put, and with non-trivial 3-form potential switched-on on their world volume (the latter accounting for the essential fifth parameter characterizing the generating solution). The compact space $\mathbb{T}_6 \times S_1$ is along directions $x^4, \ldots, x^9, x^{10}$ while the non-compact four-dimensional coordinates are x^0, x^1, x^2, x^3. The M5-branes are $N_1, N_2, N_3 q^2$, respectively, there are $N_0 + N_3 p^2$ units of KK momentum along the spatial 10th direction and the magnetic flux, related to non-trivial 3-form field strength $h_{(3)}$ excited on the M5-

M-brane cycles	Type-IIA cycles	Charges	4D Charges
KK-monopole	$D6(456789)$	0	y_0
$M5(6789\|10)$	$D4(6789)$	N_1	y_1
$M5(4589\|10)$	$D4(4589)$	N_2	y_2
$M5(4567\|10)$	$D4(4567)$	$N_3 q^2$	y_3
KK-momentum	$D0$	$N_0 + p^2\, N_3$	x_0
effective $M2(45)$	effective $D2(45)$	$-pq\, N_3$	x_1
effective $M2(67)$	effective $D2(67)$	$pq\, N_3$	x_2
effective $M2(89)$	effective $D2(89)$	0	x_3

Table 2: The effective charges along different cycles of $S_1 \times \mathbb{T}_6$ of the generating solution in terms of microscopic parameters. The signs of the different charges are the correct ones so to have a 1/8 susy preserving state. The last column gives the effective charges in terms of moduli dependent quantized charges (y^α, x_α) defined in the supergravity framework and characterizing the macroscopic description of the solution. They depend on the suitably chosen asymptotic values of the scalars fields at radial infinity, see [1] for details.

brane, is proportional to a rational number $\gamma = p/q$, $(p, q \in \mathbb{Z})$. The integers p, q are related to the angle θ characterizing the $S \times T$-dual type-IIB D3-brane configuration by the condition: $q \sin\theta = p \cos\theta$. As illustrated in [1], the solution at the asymptotically flat radial infinity is characterized by a set of eight moduli-dependent charges $(y, x) \equiv (y^\alpha, x_\alpha), \alpha = 0, \ldots, 3$ (four magnetic y^α describing the Kaluza-Klein monopole and M5 charges, and four electric x_α related to the Kaluza-Klein momentum and M2 charges).[2] The precise relation between the macroscopic charges (y, x) as related to the effective charges along the various cycles of $\mathbb{T}_6 \times S_1$ and the microscopic parameters $\{N_\alpha, p, q\}$ in the M-theory description of our solution (as well as in the S-dual type-IIA one), is given in table 2. This correspondence was worked out in [1] and was made possible thanks to an intrinsic group theoretical characterization, carried out in [2], of the ten-dimensional origin of the vector and scalar fields in the four-dimensional $N = 8$ model, once the latter is interpreted as the low energy effective theory of type-IIA/IIB superstring theories on $\mathbb{T}_6$.

The expression of the entropy in terms of the charges (y, x) is given by applying to our solution [1] (for which $y^0 = x_3 = 0$) the Bekenstein-Hawking formula and expressing the area of the horizon A_h (in suitable units) in terms of the charges at infinity:

$$S = \frac{A_h}{4 l_P^2} = 2\pi \sqrt{y_1 y_2 y_3 \left[x_0 - \frac{(x_1 y_1 - x_2 y_2)^2}{4 y_1 y_2 y_3} \right]} \tag{1.1}$$

l_P being the Plank length. The moduli dependence of the charges (y, x) in the above expression drops out, consistently with the moduli independence of the entropy.

[2] These charges are related to the moduli-independent quantized charges through a symplectic transformation (see [2] and [1]) and therefore are quantized as well.

Using table 2, the macroscopic entropy can be expressed in terms of the microscopic parameters and reads:

$$S = 2\pi \sqrt{N_1 N_2 N_3 q^2 \left[N_0 + p^2 N_3 - \frac{1}{4} p^2 N_3 \frac{(N_1 + N_2)^2}{N_1 N_2} \right]}\,, \tag{1.2}$$

where the first two terms in the square bracket correspond to x_0 (the total momentum along S_1) while the third term represents a shift Δx_0 to be subtracted to x_0 in order to define the relevant momentum contribution to the entropy and which we shall comment on in the sequel. Notice that the second and third terms in the square bracket are related to non-trivial membrane effective charges and therefore consistently vanish as $\gamma, p \to 0$ (while q can be absorbed in a re-definition of N_3), i.e. when the magnetic flux, in the type-IIA/M-theory language, or the non-trivial angle θ, in the type-IIB description, vanishes.

We wish to derive the expression (1.2) from a microscopic BPS state counting. As anticipated, this can be done along the lines of [7, 8]. Having expressed the entropy in terms of charges computed in the asymptotically flat radial infinity allows us to perform a "far from the horizon" counting of states for which the relevant framework is M-theory on $M_4 \times \mathbb{T}_7$ (or type-II superstring on $M_4 \times \mathbb{T}_6$). Although the analysis of [7, 8] is concerned with BPS black holes deriving from M-theory compactified on a manifold of the form $M_4 \times CY_3 \times S_1$ which has a different topology, we expect the low energy properties of our solution at tree level to coincide with those of the black holes studied in [7, 8] for a suitable choice of the CY_3 manifold. Indeed (at tree level) the generating solution of $N = 8$ regular BPS black holes is also a solution of an $N = 2$ consistent truncation of the $N = 8$ model, namely the STU model, whose six-dimensional scalar manifold has a geometry defined by a cubic prepotential. This model describes also the tree level low energy dynamics of black holes within M-theory on $M_4 \times CY_3 \times S_1$, where the prepotential characterizing the special Kähler geometry of the complex structure moduli space of CY_3 is cubic, at tree level. We shall check, in this particular case, that the result of the entropy counting attained in [7, 8] does coincide at tree level with the result of the analogous calculation we shall perform on our generating solution. The following analysis will be carried out in the eleven-dimensional framework, adapting to the torus the study in [7, 8]. References to the dual type-IIA setting will be done under the reasonable assumption that the degeneracy of BPS microstates is insensitive to the type-IIA/M-theory duality.[3]

Let us first briefly comment on the regime of parameters in which our computations are carried out. First of all we require the M-theory modes to decouple from the

[3] This would require in turn that, in changing the moduli of the background, singularities are not encountered and all the quantities defining BPS states behave smoothly, which is indeed the case.

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Kaluza-Klein modes of eleven-dimensional supergravity compactified on $M_4 \times \mathbb{T}_7$.[4] This happens if $\mathbb{T}_7$ is *large* in the eleven-dimensional Planck units, i.e. if $R_i \gg l_P$, R_i being the radii of the internal directions of $\mathbb{T}_7$. Secondly we demand the four-dimensional supergravity description of the generating solution to be reliable. This is the case if the curvature of the solution (whose upper bound is the near horizon curvature $\approx 1/A_h$) is smaller than the scale fixed by the Kaluza-Klein spectrum. A sufficient condition for this to hold is therefore $A_h \gg R_i^2$. Using eq. (1.1) the latter amounts to the condition that the quantized charges (x, y) be much larger than R_i/l_P ($\gg 1$).

A successful strategy for achieving a microscopic entropy counting has been so far to choose suitable limits in the background geometry such that the low energy quantum fluctuations around the solution are described by a two-dimensional conformal field theory on a torus (a cylinder in the limit of non-compact time). In this case the asymptotic value of the degeneracy of microstates $\rho(h)$ for high excited levels h is given in terms of the central charges of the σ-model by the Cardy formula [12, 13]. The conformal field theory considered in [7] emerges in the limit in which the radius of the eleventh dimension R is much larger than the radii of the remaining compact manifold, which means, in our case, $R \gg V(\mathbb{T}_6)^{1/6}$. In this limit, the low energy fluctuations of the three M5-branes will be independent of the $\mathbb{T}_6$ coordinate and described by a conformal 1+1 σ-model. On the six-dimensional world volume of each M5-brane embedded in the background of the other 2 branes there is $(0, 2)$ supersymmetry and BPS excitations will break half of them. This chiral supersymmetry will manifest on the two-dimensional conformal field theory as a $(0, 4)$ supersymmetry (this is the same effective theory describing the microscopic degrees of freedom of the configuration described in [7, 8]). The BPS states of this model will be annihilated by the right-moving supercharges and therefore they are described as excitations of the left moving sector. Denoting by c_L the left-mover central charge and by h the excitation level, the asymptotic value of $\rho(h)$ for large h ($h \gg c_L$) is given by the Cardy formula:[5]

$$\rho(h) \approx \left(\frac{c_L}{96h^3}\right)^{1/4} e^{2\pi\sqrt{c_L h/6}}\rho(h_0) \approx e^{2\pi\sqrt{c_L h/6}}\,. \tag{1.3}$$

From the above expression, using the Boltzmann formula, one can derive the asymptotic value of the black hole entropy:

$$S_{\text{micro}} = \ln \rho(h) \approx 2\pi\sqrt{\frac{c_L h}{6}}\,. \tag{1.4}$$

In the M-theory picture h is the non-zero mode contribution to the momentum along the eleventh direction S_1, i.e. in the conformal theory language, the non-zero mode

[4]Which amounts to asking Kaluza-Klein supergravity to provide a reliable description of the physics on the chosen eleven-dimensional vacuum.

[5]An other condition for the validity of this formula, which we can reasonably assume to hold in our case, is that the minimum excitation level h_0 should be small: $h_0 \ll c_L$.

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contribution to $L_0 - \bar{L}_0$. In the type-IIA description the momentum along S_1 is given by the D0-brane charge: $x_0 = N_0 + p^2 N_3$. Clearly the regimes of validity of the type-IIA picture and the description in terms of the above defined conformal field theory are opposite. Since, as previously said, the charge and density of BPS states can be supposed to be invariant with respect to the S-duality mapping large to small radius of S_1, we shall compute h, in the type-IIA framework, in terms of a suitable contribution to the D0-brane charge. This is done in section 3.

2. Computation of the central charge c_L

Let us start by computing the central charge c_L. The general expression for c_L is:

$$c_L = N_L^B + \frac{1}{2} N_L^F \tag{2.1}$$

N_L^B and N_L^F being the number of left-moving bosonic and fermionic degrees of freedom, respectively. The number of left-moving bosons has essentially two contributions. One is coming from the moduli of the 4-cycles P of the torus $\mathbb{T}_6$ along which the M5-branes are wrapped (call their number d_p). The other contribution comes from the moduli of the rank two antisymmetric tensor potential b propagating on the M5-branes world volume ($h_{(3)} = db$). We can assume that each couple of M5-branes does intersect along the common directions (e.g. the branes with charges y_1 and y_3 along the plane $(6,7)$).[6] In this case the configuration of the three M5-branes can be described by a superposition P of 4-cycles of $\mathbb{T}_6$ whose *fundamental class* is $[P] = \sum_i y^i \alpha_i$ with $\alpha_i, i = 1,2,3$ being three $(1,1)$-cycles in $H^2(\mathbb{T}_6, \mathbb{Z})$ which will be defined precisely for our configuration in the sequel[7] (recall that 2-cycles are isomorphic to 4-cocycles). As far as the first type of moduli is concerned one can see, assuming suitable conditions on $[P]$[8] and recalling that a torus has vanishing Chern classes, that:

$$d_p = \frac{1}{3} \int_{\mathbb{T}_6} [P]^3 - 2\,. \tag{2.2}$$

Let us now consider the contribution from the 2-form b. As previously stated, in the limit of small $\mathbb{T}_6$ with respect to S_1, the field b can be considered, in the low energy limit, as function of just the coordinates $\{x^{10}, x^0\}$ on $S_1 \times \mathbb{R}$. The three form $h_{(3)} = db$ defined on the M5 world volume is self-dual. This implies that among the

[6] An alternative is for the projections of the two M5-branes along the common directions to be separated by a distance along the non-compact space directions. The two configurations clearly have the same charges and energy.

[7] We take the complex structure on $\mathbb{T}_6$ to be defined in this way: $z^1 = x^4 + \mathrm{i}x^5$, $z^2 = x^6 + \mathrm{i}x^7$, $z^3 = x^8 + \mathrm{i}x^9$.

[8] Along the lines of [7], we assume the 4-cycle P to be a *very ample* divisor of $\mathbb{T}_6$, which formalizes in the language of algebraic geometry the requirement for P to be "large", besides the weaker condition for $[P]$ to define a Kähler class (positiveness), [11].

fields b which are 2-forms on P, the left-moving ones are anti-self-dual (b^-), while the right-moving ones are self-dual (b^+). Moreover, the forms b with just one index on P are non-dynamical gauge fields in the theory on $S_1 \times \mathbb{R}$ which will enter the game since $b_1 = 2h_{(1,0)}(P) \neq 0$ and on which we are going to comment in a moment. Using the Hodge index theorem on P and doing some simple calculations, one can see that the spaces of the $b^\pm$ have the following dimensions, respectively:

$$
\begin{aligned}
\dim\{b^-\} &= h_{(1,1)} - 1 = \frac{2}{3}\int_{\mathbb{T}_6}[P]^3 + 2h_{(1,0)} - 1\,, \\
\dim\{b^+\} &= 1 + 2h_{(2,0)} = \frac{1}{3}\int_{\mathbb{T}_6}[P]^3 + 2h_{(1,0)} - 1\,.
\end{aligned}
\tag{2.3}
$$

As far as the fermion modes $N^F_{L,R}$ are concerned, from standard analysis it is known that on a complex manifold the number of left-moving and right-moving fermions are related to the dimension of $H^{(2r+1,0)}(P)$ and $H^{(2r,0)}(P)$, respectively: $N^F_L = 4h_{(1,0)}$ and $N^F_R = 4(h_{(0,0)} + h_{(2,0)})$. Let us now collect our results. With the above definitions, the number of left and right moving bosons is given, in general, by:

$$
N^B_L = d_p + \dim\{b^-\} + 3\,, \qquad N^B_R = d_p + \dim\{b^+\} + 3\,, \tag{2.4}
$$

where the $+3$ in the two cases takes into account the contribution of three translational zero modes. Hence we have:

$$
\begin{aligned}
N^B_L &= \int_{\mathbb{T}_6}[P]^3 + 2h_{(1,0)}\Big\{ - 2h_{(1,0)} = \int_{\mathbb{T}_6}[P]^3 \Big\}\,, \\
N^F_L &= 4h_{(1,0)}\{-4h_{(1,0)} = 0\}\,, \\
N^B_R &= \frac{2}{3}\int_{\mathbb{T}_6}[P]^3 + 2h_{(1,0)}\Big\{ - 2h_{(1,0)} = \frac{2}{3}\int_{\mathbb{T}_6}[P]^3 \Big\}\,, \\
N^F_R &= 4h_{(2,0)} + 4\{-4h_{(1,0)}\}\,,
\end{aligned}
\tag{2.5}
$$

where the terms in the curly brackets represent the effect of the gauging of the b_1 non-dynamical gauge fields introduced previously. Indeed, coupling the (left and right moving) bosonic and fermionic modes to these vector fields will reduce the scalar degrees of freedom by $b_1 = 2h_{(1,0)}(P)$ and the fermionic ones by $2b_1$ (we require, according to [9], this coupling to be left-right symmetric). This gauging is *not* optional, since it restores supersymmetry on the right sector. This can be easily seen deriving from eq. (2.3) the following relation:

$$
\frac{1}{6}\int_{\mathbb{T}_6}[P]^3 = h_{(2,0)} - h_{(1,0)} + 1 \tag{2.6}
$$

which in turn implies that, provided the gauging is performed, the values of N^B_R and N^F_R, read off from eqs. (2.5), coincide. From eq. (2.1) and eqs. (2.5) we can finally deduce the value of the central charge c_L to be: $c_L = \int_{\mathbb{T}_6}[P]^3$. To compute its value in

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terms of the quantized charges y^i we consider suitable representatives of the classes α_i and define as D_{ijk} the restriction to them of the triple intersection numbers of $\mathbb{T}_6$:

$$1 = 6D_{ijk} = \int \alpha_i \wedge \alpha_j \wedge \alpha_k\,, \qquad \alpha_i = dx^a \wedge dx^b\,,$$
$$\{i\} = \{1,2,3\} \equiv \{(ab)\} = \{(45),(67),(89)\}\,. \tag{2.7}$$

The left-mover central charge is therefore:

$$c_L = \int_{\mathbb{T}_6} [P]^3 = 6y_1y_2y_3 = 6q^2N_1N_2N_3\,. \tag{2.8}$$

Notice that the gauging discussed earlier besides ensuring supersymmetry on the right-mover sector, provides a consistency condition for the left sector as well. Indeed if it were not performed, c_L would have an additional term proportional to b_1. As a consequence of this, the entropy computed from eq. (1.4), provided $h \neq 0$, would have a non-vanishing leading contribution even if the number of intersecting (bunches of) branes would have been less than three,[9] in disagreement with the Bekenstein-Hawking macroscopic prediction (differently, in the case of a generic C-Y manifold one needs instead just one set of coinciding branes to have a regular horizon in four dimensions, see for instance [14]).

3. Non-zero mode contribution to the momentum along S_1: a type-IIA computation

The remaining quantity to be computed on our solution is the excitation level h in eq. (1.4). This can be done by extending to our configuration on $\mathbb{T}_6$ the computation performed in [7] within the M-theory framework. As previously anticipated, we shall adopt the type-IIA viewpoint instead, and write h as the difference between the *total* D0-brane charge $x_0 = N_0 + p^2N_3$ related to the background configuration and a contribution Δx_0 to be suitably interpreted from type-IIA perspective. While the former quantity corresponds in eleven dimensions to the total KK momentum along x^{10}, i.e. in the CFT limit, to the eigenvalue of $L_0 - \bar{L}_0$, the latter defines the zero-mode contribution to $L_0 - \bar{L}_0$. On the type-IIA side Δx_0, as we shall see, can be expressed in terms of the contribution to the D0-brane charge due to a magnetic flux $\mathcal{F}^{(0)}$, defined on the intersections along the planes (45), (67), (89) of the D4-branes, which can be interpreted as the *part* of the total magnetic flux $\mathcal{F}$ due to the same modes or states which in the S-dual CFT picture are zero-modes of the potential b. In the language of open strings attached to D4-branes these states can be possibly

[9]As explained above we are considering the limit where $N_0 \gg c_L$. In this regime, the dominant configuration is that of "short" branes [15]: we have three bunches of parallel M5-branes, N_1, N_2, N_3q^2, respectively which therefore intersect on $N_1N_2N_3q^2$ points on $\mathbb{T}_6$.

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identified with modes of Dirichelet-Dirichelet strings connecting the $N_3 q^2$ branes to the N_1 and N_2 branes, which are massless and will induce an equal flux density $\mathcal{F}^{(0)}$ on the world volume of the N_i and N_j branes along their common directions. The flux $\mathcal{F}^{(0)}$ will correspond, upon dimensional oxidation and suitable dualization, to a 3-form $h^{(0)}_{(3)}$ on the M5-branes. The derivation of the WZ term in the world volume action of the D4-brane, yielding the contribution of the magnetic flux to the D0-brane charge, from the corresponding term in the M5-brane world volume action, is discussed in the appendix, and the result can be applied to the fields $\mathcal{F}^{(0)}$ and $h^{(0)}_{(3)}$.

The magnetic flux density $\mathcal{F}^{(0)}$, differently from the whole $\mathcal{F}$ which is non vanishing only on the world volume of the $N_3 q^2$ branes, can be defined on $\mathbb{T}_6$ in terms of the three 2-forms α_i: $\mathcal{F}^{(0)} = \sum_i \mathcal{F}^{(0)|i}\alpha_i$.[10] Its value is determined in terms of the effective electric charges x_i through the WZ terms which couple it to the D4-brane world volumes. The part of the R-R 3-form A coupled to $\mathcal{F}^{(0)}$ in these terms is $\sum_i A^i_\mu dx^\mu \wedge \alpha_i$, where the index μ runs along the non-compact directions and the vectors fields A^i_μ denote three electric potentials of the effective four-dimensional theory. The corresponding charges x_i are defined by the four-dimensional minimal couplings:

$$\begin{aligned} x_i \int A^i_0 dx^0 &= \frac{1}{2\pi}\sum_k \int_{(D4)_k} \mathcal{F}^{(0)|j}\alpha_j \wedge \alpha_i \wedge A^i_0 dx^0 \\ &= \left(6D_{ijk}y^k \frac{\mathcal{F}^{(0)|j}}{2\pi}\right) \times \int A^i_0 dx^0 \end{aligned} \tag{3.1}$$

hence:

$$x_i = 6D_{ijk}y^j G^k = 6D_{ij}G^j\,, \tag{3.2}$$

where we have defined $G^i = F^{(0)|i}/(2\pi)$ and $D_{ij} = D_{ijk}y^k = \int_P \alpha_i \wedge \alpha_j/6$. The integrals on the right hand side of eq. (3.1) are computed on the D4-brane world volumes, and eventually extended to the whole $\mathbb{T}_6$ using the fundamental class $[P]$ previously defined. We have considered moreover $\mathcal{F}^{(0)|i}$ to be uniform along the planes on which it is defined. The contribution of $\mathcal{F}^{(0)}$ to the total D0-brane charge is given by:

$$\Delta x_0 = -\frac{1}{2(2\pi)^2}\int_P \mathcal{F}^{(0)} \wedge \mathcal{F}^{(0)} = 3D_{ij}G^i G^j\,. \tag{3.3}$$

Inverting eq. (3.2) one finds $G^i = D^{ij}x_j/6$, where $D^{ij}D_{jk} = \delta^i_k$. Inserting this result in eq. (3.3) we obtain:

$$\Delta x_0 = \frac{1}{4y_1 y_2 y_3}\left(\sum_i (y_i x_i)^2 - 2\sum_{i<j} y_i x_i y_j x_j\right) = \frac{p^2 N_3}{4N_1 N_2}(N_1+N_2)^2\,, \tag{3.4}$$

where in the last passage we have used the expression of the charges (x, y) given in table 2. This quantity, being related in the S-dual CFT picture, to the zero-mode

[10]The components $\{\mathcal{F}^{(0)|1}, \mathcal{F}^{(0)|2}, \mathcal{F}^{(0)|3}\}$ correspond to $\{\mathcal{F}^{(0)}_{45}, \mathcal{F}^{(0)}_{67}, \mathcal{F}^{(0)}_{89}\}$, consistently with the convention on the indices defined in eq. (2.7).

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contribution to the KK momentum along S_1, has to be subtracted from the total D0-brane charge x_0 in order to obtain h. Hence we finally get:

$$h = x_0 - \Delta x_0 = N_0 + p^2 N_3 - \frac{p^2 N_3}{4 N_1 N_2}(N_1 + N_2)^2 \,. \tag{3.5}$$

Inserting the above expression and eq. (2.8) in eq. (1.4) one finally obtains the correct microscopic prediction for the Bekenstein-Hawking entropy, eq. (1.2)! This ends the computation.

4. Conclusion

Our final formula is the same as (3.28) of [7]. However we wish to emphasize that the above computed microscopic entropy formula accounts for the entropy of *all* regular BPS black holes within toroidally compactified string (or M) theory, [1]. In particular, the *shift* factor encodes the essential fifth parameter which is crucial for the solution to be a generating one. Notice that since the near horizon geometry is characterized by just one of the 5 parameters (i.e. the entropy or horizon area) which is the one accounting for the microscopic degeneracy of the (proper) black hole states,[11] we actually expect that this geometry does not distinguish between regular solutions characterized by different numbers of U-duality invariants. Indeed, for suitable values of the corresponding quantized charges, two different solutions can have precisely the same near horizon geometry (and therefore the same entropy). For instance, the difference between the expression of the entropy for a five or a four parameter solution may amount just to the aforementioned shift which can be absorbed in a re-definition of the quantized charges (as far as near horizon geometry is concerned). However, if we wish to characterize the entropy of a (proper) regular black hole as part of the most general interpolating solution, then the parameter characterizing this shift is an essential ingredient for supporting all the independent U-duality invariant degrees of freedom. Therefore it is far from the horizon where this shift parameter acquires a highly non-trivial physical meaning. The magnetic flux manifests itself by coupling to new scalars (the axions) whose radial evolution in the solution is non trivial. These extra scalars are needed for the solution to be a generating one, [1].

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[11] Here by "proper" black hole we mean the near horizon solution which is *hairless* thanks to supersymmetry, as opposite to the whole solution interpolating between the horizon and the asymptotically flat radial infinity, which has indeed *hair*, i.e. it depends on matter fields. The microscopic degrees of freedom of this *hair* are encoded in four of the five U-duality invariants.

A. D0-brane effective charge from eleven dimensions

Let us briefly review the derivation of the WZ term describing the contribution to the D0-brane charge due to a magnetic flux $\mathcal{F}$ on the D4-brane from the perspective of M-theory compactified on a circle. The normalization we choose for this term in the low energy effective action on the D4-brane is the following:

$$-\frac{1}{2(2\pi)^2}\int_P \mathcal{F}\wedge\mathcal{F}\int C_{(1)}\,, \tag{A.1}$$

where $C_{(1)}$ is the pull-back of the R-R 1-form which couples to the D0-brane and P is the four cycle of $\mathbb{T}_6$ on which the D4-brane is wrapped.

In general, the derivation of the D4-brane action from that of the M5-brane upon double dimensional reduction is not straightforward. In fact, as shown in detail in [16, 17], by compactifying the world volume action of the M5-brane one ends in an dual gauge. In order to get the usual D4-brane action one has to perform a (electro-megnetic) duality transformation, which moreover mixes the DBI and the WZ terms. Here, however, we are just interested in deriving the term (A.1), and computations can be much simplified. If we take the background gauge potential to have only the temporal component non-vanishing and choose static gauge, then only C_0 survives on the world volume (and coincides with the corresponding component of the background field). If we now choose it to be small, the WZ term (A.1) turns out to be the same in the two dual regimes, at leading order. This can be easily seen by comparison of formulae (81) and (82) of the appendix of [17].

Let us then consider the theory on the M5-brane (double oxidation of the D4-brane) embedded in a Minkowsky space-time $G_{MN}=\eta_{MN}$ $(M,N=0,\dots,10)$ and perform an infinitesimal shift on this metric by a quantity δG_{MN} whose only non-zero entry is $\delta G_{010}=C_0\ll 1$ (we are thinking of the dimensional reduction on S_1 to ten dimensions and of the known fact that ultimately $C_m\equiv\delta G_{10m}$). The world volume action of the M5-brane would acquire a term of the form:

$$\begin{aligned}\delta S &= -\int_{S_1\times\mathbb{R}} T_{vw}\delta\hat{G}^{vw}\,d\xi^{10}d\xi^{0} = -2\int_{S_1\times\mathbb{R}} T_{010}C_0\,d\xi^{10}d\xi^{0}\,,\\ T_{010} &= \beta\left({h_0}^{ab}h_{10cd}\right)\int_P \alpha_{ab}\wedge{}^*\alpha^{cd}\propto h_{0ab}{h_{10}}^{ab}\,,\end{aligned} \tag{A.2}$$

where it is understood that the M5-brane world volume extends over $P\times S_1\times\mathbb{R}$, the indexes v,w get values $0,\dots,6,10$, $\hat{G}$ is the pull-back of the metric G, and $*$ is the Hodge duality on P. The constant β is determined by our choice of normalization in eq. (A.1) and will be fixed at the end. If we label by a the internal directions of $\mathbb{T}_6$ then we choose the components h_{abc} of the self-dual tensor to be zero. The self-duality of $h_{(3)}$ implies that ${h_\pm}^{ab}\alpha^{\mp}_{ab}=0$, where "$\pm$" labels on the $h_{(3)}$ tensor the light-cone coordinates $\xi^\pm = \xi^0 \pm \xi^{10}$ and on the 2-form α its self-dual/anti-self-dual components in P: $\alpha = \alpha^+ + \alpha^-$, ${}^*\alpha = \alpha^+ - \alpha^-$.

We may rewrite the action term in eq. (A.2) as follows:

$$\begin{aligned} \delta S &= -2\beta \int_{P\times S_1} {h_0}^{ab} h_{10cd} \alpha_{ab} \wedge {}^*\alpha^{cd} \wedge d\xi^{10} \int C_0 d\xi^0 \\ &= -2\beta \int_{P\times S_1} \left({h_+}^{ab} h_{+cd} - {h_-}^{ab} h_{-cd}\right) \alpha_{ab} \wedge {}^*\alpha^{cd} \wedge d\xi^{10} \int C_0 d\xi^0 \\ &= -2\beta \int_{P\times S_1} {h_{10}}^{ab} {h_{10}}^{cd} \alpha_{ab} \wedge \alpha_{cd} \wedge d\xi^{10} \int C_0 d\xi^0 \,, \end{aligned} \tag{A.3}$$

where we have used the self-duality of $h_{(3)}$.

If we integrate the expression in eq. (A.3) over S_1 (very small), after considering only the zero-modes along it, and make the (leading-order) identification $\mathcal{F}_{ab}/(2\pi) = h_{10ab}$ (see [18]) we obtain precisely the term in eq. (A.1) setting $\beta = 1/(8\pi R)$, R being the radius of the eleventh dimension.

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